Exercise 5

1. Find the partial derivatives of the following functions:

   (a) $(xy - 5z)/(1 + x^2)$,
   (b) $x/\sqrt{x^2 + y^2}$,
   (c) $\arctan y/x$,
   (d) $\log((t + 1)^3 + ts^2)$,
   (e) $\sin(xy^2z^3)$,
   (f) $|x|^\alpha, \quad x = (x_1, \cdots, x_n)$.

2. Verify $f_{xy} = f_{yx}$ for the following functions:

   (a) $x \cos y + e^{2y}$,
   (b) $x \log(1 + y^2) - \sin(xy)$,
   (c) $(x + y)/(x^5 - y^9)$.

3. Consider the function

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0) ,$$

   and $f(0, 0) = 0$. Show that f_{xy} and f_{yx} exist but are not equal at $(0, 0)$.

4. Find

$$\frac{\partial^3 u}{\partial x \partial y \partial z} , \qquad \text{where } u(x, y, z) = e^{xyz} .$$

5. * Show that

$$\frac{\partial^{m+n} v}{\partial x^m \partial y^n} = \frac{2(-1)^m (m + n - 1)!(mx + ny)}{(x - y)^{m+n+1}} ,$$

   where

$$v(x, y) = \frac{x + y}{x - y} .$$

6. *

   (a) A harmonic function is a function satisfies the Laplace equation

$$\Delta u \equiv \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n} \right) u = 0 .$$

   Show that all n-dimensional harmonic functions form a vector space.

(b) Find all harmonic functions which are polynomials of degree ≤ 2 for the two dimensional Laplace equations. Show that they form a subspace and determine its dimension.

7. Consider the function
$$g(x,y) = \sqrt{|xy|}\ .$$
Show that g_x and g_y exist but g is not differentiable at $(0,0)$.

8. Consider the function $h(x,y) = 1$ for (x,y) satisfying $x^2 < y < 4x^2$ and $h(x,y) = 0$ otherwise. Show that h_x and h_y exist but h is not differentiable at $(0,0)$.

9. Consider the function $j(x,y) = (x^2+y^2)\sin(x^2+y^2)^{-1}$ for $(x,y) \neq (0,0)$ and $j(0,0) = 0$. Show that it is differentiable at $(0,0)$ but its partial derivatives are not continuous there.

10. Use the Chain Rule to compute the first and second derivatives of the following functions.

   (a) $f(x+y, x-y)$,
   (b) $g(x/y, y/z)$,
   (c) $h(t, t^2, t^3)$,
   (d) $f(r\cos\theta, r\sin\theta)$,

11. * Let $f(x,y)$ and $\varphi(x)$ be continuously differentiable functions and define
$$G(x) = \int_0^{\varphi(x)} f(x,y)dy\ .$$
Establish the formula
$$G'(x) = \int_0^{\varphi(x)} f_x(x,y)dy + f(x,\varphi(x))\varphi'(x)\ .$$
Hint: Consider the function
$$F(x,t) = \int_0^t f(x,y)dy\ .$$

12. (a) Show that the ordinary differential equation satisfied by the solution of the Laplace equation in two dimension $\Delta u = 0$ when u depends only on the radius, that is,
$$u = f(r), \quad r = \sqrt{x^2+y^2}\ ,$$
is given by
$$f''(r) + \frac{1}{r}f'(r) = 0\ .$$
   (b) Can you find all these radially symmetric harmonic functions?

13. (a) Show that the ordinary differential equation satisfied by the solution of the Laplace equation in three dimension $\Delta u = 0$ when u depends only on the radius, that is,

$$u = f(r), \quad r = \sqrt{x^2 + y^2 + z^2} \ ,$$

is given by

$$f''(r) + \frac{2}{r} f'(r) = 0 \ .$$

(b) Can you find all these radially symmetric harmonic functions?

14. Consider the one dimensional heat equation

$$u_t = u_{xx} \ .$$

(a) Show that $u(x,t) = v(y), y = x/\sqrt{t}$, solves this equation whenever v satisfies

$$v_{yy} + \frac{1}{2} y v_y = 0 \ .$$

(b) Show that $u(x,t) = e^{xt+2t^3/3} w(y), y = x+t^2$, solves this equation whenever w satisfies

$$w_{yy} = yw \ .$$

15. * Let u be a solution to the two dimensional Laplace equation. Show that the function

$$v(x,y) = u\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

also solves the same equation. Hint: Use $\Delta \log r = 0$ where $r = \sqrt{x^2+y^2}$.

16. * Express the differential equation

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0 \ ,$$

in the new variables

$$\xi = x, \quad \eta = x^2 + y^2.$$

Can you solve it?

17. Express the one dimensional wave equation

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0 \ , \quad c > 0 \text{ a constant} \ ,$$

in the new variables

$$\xi = x - ct, \quad \eta = x + ct \ .$$

Then show that the general solution to this equation is

$$f(x,y) = \varphi(x - ct) + \psi(x + ct) \ ,$$

where φ and ψ are two arbitrary twice differentiable functions on $\mathbb{R}$.

18. Consider the Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} + rxV_x - rV = 0 \ .$$

(a) Show that by setting $V(x,t) = u(y,t),\ y = \log x, \sigma^2 t = -2\tau$, the equation is turned into

$$-w_\tau + w_{yy} + \frac{2r}{\sigma^2}(w_y - w) = 0 \ .$$

(b) Show that further by setting $w(y,\tau) = e^{\alpha y + \beta\tau} u(y,\tau)$, with suitable α and β, the equation becomes the heat equation

$$u_\tau - u_{yy} = 0 \ .$$

19. A polynomial P is called a homogeneous polynomial if all terms have the same combined power, that is, there is some m such that $P(tx) = t^m P(x)$ for all $t > 0$. Establish the Euler's identity

$$\sum_{j=1}^{n} x_j \frac{\partial P}{\partial x_j} = P(x) \ .$$

Verify it for the following homogeneous polynomials:

(a) $x^2 - 3xy + y^2$, and

(b) $x^{15} - x^{10}y^3z^2 + 6y^{14}z$.

20. An open set D is called connected if for every $x, y \in D$, there exists a parametric curve lying in D connecting x and y. Show that a differentiable function f in an open, connected set with vanishing partial derivatives must be a constant. Hint: Use a regular parametric curve to connect x to y and consider the composite of this curve with f. Chain Rule will do the rest.

21. Find the directional derivative of each of the following functions at the given point and direction:

(a) $x^2 + y^3 + z^4, \quad (3,2,1);\ (-1,0,4)/\sqrt{17}$.

(b) $e^{xy} + \sin(x^2 + y^2), \quad (1,-3);\ (1,1)/\sqrt{2}$.

22. Find the directional derivative of the function $x^2 - y^2$ at $(1,1)$ whose direction makes an angle of degree 60° with the x-axis.

23. Let $g(x,y) = x^2 - xy + y^2$. Find

(a) the direction along which it increases most rapidly.

(b) the direction along which it decreases most rapidly.

(c) the directional at which its directional derivative vanishes.

24. Can you find a function whose directional derivative along every direction exists and all equal at $(0,0)$ but it is not differentiable there ? Hint: An example can be found in a previous problem.

25. (a) Let $f(x,y)$ be a function defined in the first quadrant $\{(x,y):\ x,y \geq 0\}$. Propose a definition of the partial derivatives of f at $(x,0), x>0$ and at $(0,0)$.

    (b) Let $g(x,y)$ be a function defined in the set $\{(x,y):\ 0 \leq x \leq y\}$. Propose a definition of the partial derivatives of g at $(0,0)$.

26. Use the differential of an appropriate function to obtain an approximate error estimate and then compare it with the actual one. You may use a calculator.

    (a) $\sin 29^\circ \times \tan 46^\circ$.

    (b) $\dfrac{1.03^2}{(0.98)^{1/3}(1.05)^{3/4}}$.

    (c) $\sqrt{(3.1)^2+(4.2)^2+(11.7)^2}$.

27. The height and the radius of the base of a cylinder are measured with error up to 0.1 and 0.2 respectively. Find the approximate and exact maximum error of its volume.

28. A horizontal beam is supported at both ends and supports a uniform load. The deflection at its midpoint is given by
$$S = \frac{k}{wh^3} ,$$
where w and h are the width and height respectively of the beam and k is some constant depending on the beam. Show that
$$dS = -S\Big(\frac{1}{w}dw + \frac{3}{h}dh\Big) .$$
If $S = 1$ in. when $w = 2$ in. and $h = 4$ in., approximate the deflection when $w = 2.1$ in. and $h = 4.1$ in.. Then compare your approximation with the actual value.

29. The point $(1,2)$ lies on the curve defined by the equation
$$f(x,y) = 2x^3 + y^3 - 5xy = 0 .$$
Approximate the y-coordinate of the nearby point (x,y) on this curve which $x = 1.2$.

30. Suppose that $T = x(e^y + e^{-y})$ where $x = 2, y = \log 2$ with maximum possible errors 0.1 in x and 0.02 in y. Estimate the maximum possible induced error in the computed value of T.